1.

Equations of the second Order

Change of Dependent variable

A linear equation of the second order has the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \text{--- (1)}$$

Where P, Q, R are functions of x or constants

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ --- (2)

Suppose $y = u$ (Where u is some function of x) satisfies the equation (2).

$$\therefore \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \quad \text{--- (3)}$$

Let $y = u \cdot v$, Where v is also function of x.

Then,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

and $\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

putting the values of $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in (1), we have

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$u\frac{d^2v}{dx^2}+2\frac{du}{dx}\cdot\frac{dv}{dx}+v\frac{d^2u}{dx^2}$$

$$+P\left\{u\frac{dv}{dx}+v\frac{du}{dx}\right\}+Q\cdot uv=R$$

$$\Rightarrow u\frac{d^2v}{dx^2}+\left(2\frac{du}{dx}+Pu\right)\frac{dv}{dx}$$

$$+\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)v=R$$

$$\therefore u\frac{d^2v}{dx^2}+\left(2\frac{du}{dx}+Pu\right)\frac{dv}{dx}=R$$

$$\Rightarrow \frac{d^2v}{dx^2}+P_1\frac{dv}{dx}=R_1 \quad \text{——— } (4)$$

Where $P_1=\frac{2}{u}\frac{du}{dx}+P$ & $R_1=\frac{R}{u}$

putting $\frac{dv}{dx}=p$ we have, from (4)

$$\frac{dp}{dx}+P_1p=R_1 \quad \text{——— } (5)$$

It is a linear equation in p of first orde

Where $I.F.=e^{\int P_1dx}$

$$=e^{\int\left(\frac{2}{u}\cdot\frac{du}{dx}+P\right)dx}$$

$$=e^{\int\frac{2}{u}du}\cdot e^{\int Pdx}$$

$$I.F.=u^2e^{\int Pdx}$$

equation (5) becomes (on multiplying I.F.)

$$\frac{d}{dx}\left\{p.u^2 e^{\int P dx}\right\} = \frac{R}{u} u^2 e^{\int P dx}$$

$$\Rightarrow u^2 p e^{\int P dx} = A + \int uR e^{\int P dx} dx$$

$$\because p = \frac{dv}{dx}, \text{ we have}$$

$$v = B + A\int \frac{1}{u^2} e^{-\int P dx}. dx + \int\left\{\frac{1}{u^2} e^{-\int P dx}\int uRe^{\int P dx}. dx\right\} dx$$

Hence the complete primitive of (1) is

$$\because y = u.v$$

$$\Rightarrow y = Bu + Au\int \frac{1}{u^2} e^{-\int P dx}. dx$$

$$+ u\int\left\{\frac{1}{u^2} e^{-\int P dx}\int uR e^{\int P dx}. dx\right\} dx.$$

Where A & B are two arbitrary constants.

Remark:→

(1) If $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ —— (1)

If $y = e^{mx}$, then $\frac{dy}{dx} = me^{mx}$ & $\frac{d^2y}{dx^2} = m^2e^{mx}$

$$\Rightarrow m^2 \cdot e^{mx} + P \cdot m e^{mx} + Q e^{mx} = 0$$

$$\Rightarrow (m^2 + Pm + Q) e^{mx} = 0$$

$$\Rightarrow \boxed{m^2 + Pm + Q = 0}$$

(2) If $y = x^m$, then

$$\frac{dy}{dx} = mx^{m-1} \text{ and } \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

putting these values in (1), we have

$$m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$$

$$\Rightarrow \{m(m-1) + Pmx + Qx^2\} x^{m-2} = 0$$

$$\Rightarrow \boxed{m(m-1) + Pmx + Qx^2 = 0}$$

Note: → For the equation $(D^2+PD+Q)y=0$, where $D \equiv \frac{d}{dx}$.

(i) $y=x$ is a particular solution if $\underline{P+Qx=0}$

(ii) $y=x^2$ is a particular solution if $2+2Px+Qx^2=0$

(iii) $y=x^m$ is a particular solution if

$$m(m-1)+Pmx+Qx^2=0.$$

(iv) $y=e^{mx}$ is a particular solution if $m^2+mP+Q=0$

(v) $y=e^x$ is a particular solution if $1+P+Q=0$

(vi) $y=e^{-x}$ is a particular solution if $1-P+Q=0$

Example (1) Verify that x is a solution of the reduced equation

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=x^2 \quad \text{—} (1)$$

Solve the equation after reducing it to a first order linear equation.

Solution: →

The reduced equation of the given equation is

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0 \quad \text{—} (2)$$

~~Where~~ To verify that $y=x$ is a Solution of this reduced equation.

Now, $\frac{dy}{dx}=1$ & $\frac{d^2y}{dx^2}=0$

$$\therefore x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y = 0-x\cdot 1+x=0$$

$\Rightarrow y=x$ is a solution of eq^n (2)

Let $y = vx$, where v is a function of x.

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

and $$\frac{d^2y}{dx^2} = \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

putting these values in equation (1), we get

$$\because \quad x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^2$$

$$\Rightarrow x^3\frac{d^2v}{dx^2} + 2x^2\frac{dv}{dx} - \cancel{v/x} - x^2 \cdot \frac{dv}{dx} + \cancel{v/x} = x^2$$

$$\Rightarrow x^3\frac{d^2v}{dx^2} + x^2\frac{dv}{dx} = x^2$$

$$\therefore \quad x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 1$$

put $\frac{dv}{dx} = p$ & let $x \neq 0$, then

$$x\frac{dp}{dx} + p = 1 \qquad \text{——— (3)}$$

$$\Rightarrow \frac{dp}{dx} + \frac{1}{x}p = \frac{1}{x}$$

Which is a linear equation of first order.

$$\therefore I.F. = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Solution of (3) is

$$px = \int \frac{1}{x} \cdot x \, dx = x + c_1$$

$$\Rightarrow \quad p = 1 + \frac{c_1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx} = 1 + \frac{c_1}{x}$$

$$\therefore \quad dv = \left(1 + \frac{c_1}{x}\right) dx$$

integrating we have.

$$v = x + c_1 \log x + c_2$$

$$\Rightarrow \quad \frac{y}{x} = x + c_1 \log x + c_2$$

$$\therefore \quad y = x^2 + c_1 x \log x + c_2 x$$

Ans.

Example (2) solve

$$x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3 \quad -(1)$$

Solution:-

$$\therefore \frac{d^2y}{dx^2} - \frac{2}{x}(1+x)\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x$$

Where $P = -\frac{2}{x}(1+x)$, $Q = \frac{2(1+x)}{x^2}$

$$\because \quad P + Qx = 0$$

$\therefore$ $y = x$ is a particular solⁿ of

$$x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = 0$$

Let $y = vx$, where v is a function of x.

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \& \quad \frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

putting these values in (1), we have

$$\because \frac{d^2y}{dx^2} - \frac{2}{x}(1+x)\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x$$

$$\Rightarrow x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2}{x}(1+x)\left\{v + x\frac{dv}{dx}\right\} + \frac{2(1+x)}{x^2}.vx = x$$

$$\Rightarrow x\frac{d^2v}{dx^2} + \{2 - 2(1+x)\}\frac{dv}{dx} = x$$

$$\Rightarrow x\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} = x$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \quad \text{---} (2)$$

putting $p = \frac{dv}{dx} \Rightarrow \frac{dp}{dx} - 2p = 1$

$$\therefore I.F. = e^{\int -2dx} = e^{-2x}.$$

$\therefore$ solⁿ of (2) is

$$p.e^{-2x} = \int e^{-2x}dx = -\frac{1}{2}e^{-2x} + A$$

$$\Rightarrow p = \frac{dv}{dx} = Ae^{2x} - \frac{1}{2}$$

$$\Rightarrow v = \int\left(Ae^{2x} - \frac{1}{2}\right)dx = \frac{1}{2}Ae^{2x} - \frac{1}{2}x + \frac{1}{2}B$$

$$\Rightarrow \frac{y}{x} = \frac{1}{2}Ae^{2x} - \frac{1}{2}x + \frac{1}{2}B$$

$$\therefore y = \frac{1}{2}x\left(Ae^{2x} + B - x\right)$$

Ans.